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COMMENT

Comment on 'Generalized Bessel functions in tunnelling ionization'

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Abstract

In the recent paper Reiss and Krainov (2003 *J. Phys. A: Math. Gen.* **36** 5575) developed two new approximations for the generalized Bessel function that arises in the analytical treatment of strong-field multiphoton ionization theories. We show that their 'tunnelling approximation 1' systematically underestimates the theoretical ionization rate by a factor of 2 or more for typical laser frequencies. More important still, only a few first ATI peaks of the lowest energy may be reproduced quite well using this approximation.

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The generalized Bessel function $J_n(u, v) = \sum_{k=-\infty}^{\infty} J_{n-2k}(u)J_k(v)$ [1, 2] appears in various formulae describing ionization (or detachment) rates of atoms (or ions) in intense linearly polarized laser field. One of these expressions is given below in equation (1). The formula (1) has a form of infinite sum over n-photon processes, and partial ionization rates Γ_n are the heights of ATI (above-threshold ionization [3]) succeeding δ-function energy peaks, which are separated by the single-photon energy. In what follows we use atomic units: $\hbar = e = m_e = 1$.

$$\Gamma = \sum_{n=n_0}^{\infty} \Gamma_n = 2\sqrt{2} \sum_{n=n_0}^{\infty} \frac{\sqrt{E_n}}{\left(E_n + \frac{1}{2}\right)^2} \int_0^{\pi} J_n^2\left(\sqrt{\frac{8zE_n}{\omega}}\cos\vartheta, -\frac{z}{2}\right)\sin\vartheta \,\mathrm{d}\vartheta \tag{1}$$

This is the SFA (strong-field approximation) [2] nonrelativistic result for the 1S hydrogen atom in a linearly polarized plane-wave field of frequency ω and intensity $I = 4z\omega^3$. The integral over ϑ reflects summation over various directions of the ionized outgoing electron. The kinetic energy of this electron is $E_n = \frac{p_n^2}{2} = (n - z)\omega - E_B$, the binding energy of the 1S state is $E_B = \frac{1}{2}$, and the minimal number of photons absorbed is $n_0 = \left[\frac{E_B}{\omega} + z\right] + 1$, where the symbol $[\cdots]$ denotes the integer part of the (positive) number inside. The intensity parameter z is connected with $U_P = z\omega = \frac{I}{4\omega^2}$ where U_P stands for the ponderomotive potential (the time-averaged kinetic energy of a classical free charge in an electromagnetic plane-wave field), and I stands for the radiation intensity in atomic units (1 au $= 3.51 \times 10^{16}$ W cm^{-2}). In the SFA

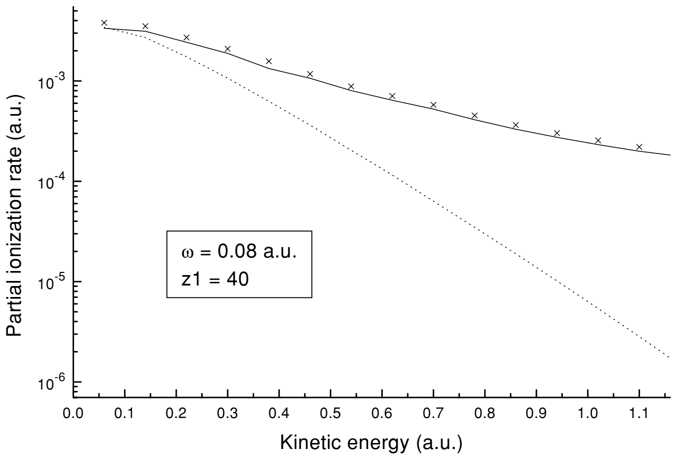

Figure 1. Partial ionization rate Γ_n as a function of the kinetic energy E_n of the outgoing electron (see equation (1)). Theoretical SFA calculations were done for the ground state of the hydrogen atom by a linearly polarized plane wave laser field of frequency $\omega = 0.08$ au and intensity given by the parameter $z1 = 40$; solid line: with the exact generalized Bessel functions, dotted line: with the 'tunnelling 1' approximation of the generalized Bessel functions, and crosses: with the asymptotic approximation of the generalized Bessel functions.

one assumes [2] that the laser field is strong enough that one can neglect the influence of the Coulomb potential on the final state of the outgoing electron. This may be true only when the oscillation energy of the ionized electron in the laser field dominates the atomic binding energy. Therefore the dimensionless parameter $z1 \equiv 2U_P/E_B = 1/\kappa^2$ should be much larger than unity (κ is the Keldysh parameter [4]).

If the frequency of incoming electromagnetic field is low enough (like for light interacting with the ground state of the hydrogen atom) photoionization may be understood, at least to some extent, as quasi-static tunnelling. One often determines the tunnelling regime by the conditions $\kappa \ll 1$ and $\omega \ll 1$. The potential barrier, which is penetrated by a bound atomic electron, arises as the combined effect of the atomic Coulomb potential and the potential of slowly varying electric field of the plane wave [3]. The peak of the barrier lies above the 1S level only if $F < 1/16$ (F: the electric field amplitude of the laser). In the opposite case, for stronger fields, instead of tunnelling, barrier-suppression (or above-barrier) ionization takes place [5]. The SFA (or in general the nonperturbative KFR theory [4, 6, 2]) treats both these domains on equal footing in the multiphoton picture, where the ionized electron may absorb $n = n_0, n_0 + 1, n_0 + 2, \ldots$ photons. In [1] Reiss and Krainov claim that 'Tunnelling occurs with low velocity of the ionized electron, so $n \approx n_0$', and then they consider only the electrons with $n_0 \gg 1$, $n - n_0 \leqslant 10$.

In figures 1 and 2 we show the ATI spectra of electrons from the SFA theory of 1S hydrogen atom for two different frequencies of incoming radiation. We have calculated these spectra using equation (1) with the exact generalized Bessel function (solid line), the 'tunnelling 1' approximation [1] of the generalized Bessel function (dotted line), and the asymptotic approximation [1, 2] of this function (crosses). Of course, the lines are envelopes of the energy peaks successively separated by the photon energy ω. In figure 1 we have chosen $\omega = 0.08$ au, because this frequency is roughly in the middle of the optical spectrum, and $z1 = 40$, because it is well in the SFA applicability range. For these parameters the electric field amplitude is $F = 0.506 > 1/16$, so actually above-barrier ionization takes place.

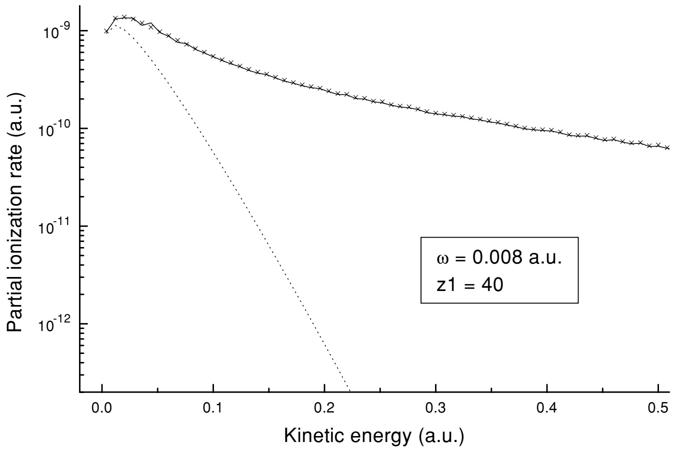

Figure 2. As figure 1, but for a lower frequency $\omega = 0.008$.

In figure 2 we have taken ten times smaller frequency $\omega = 0.008$ au, and the same $z1$, hence the electric field amplitude is now $F = 0.0506 < 0.0625 = 1/16$. Therefore one might expect the 'tunnelling 1' approximation to be more suitable in this case.

While the tunnelling conditions expressed in equations (3)–(5) of Reiss are correct, the assumption that only the lowest energy peaks are dominant in strong-field ionization is very weakly satisfied. The partial ionization rates Γ_n are largest for $n \approx n_0$ indeed, but there are much more following energy peaks, whose total contribution to ionization rate is even greater or at least quite significant. One must remember that the SFA energy spectrum of ionized electrons is infinite. In figures 1, 2 we have cut the spectra in such a way that an average kinetic energy $\langle E \rangle$ of the outgoing electron lies in the middle of the abscissa axis. From the expression $\langle E \rangle = \sum_{n=n_0}^{\infty} \Gamma_n E_n \Big/ \sum_{n=n_0}^{\infty} \Gamma_n$, we have obtained $\langle E \rangle = 0.58$ for $\omega = 0.08$ and $\langle E \rangle = 0.26$ for $\omega = 0.008$ using the exact generalized Bessel functions. In the 'tunnelling 1' approximation respective values are much smaller: $\langle E \rangle = 0.19$ for $\omega = 0.08$ and $\langle E \rangle = 0.032$ for $\omega = 0.008$. We clearly see that the 'tunnelling 1' approximation fails rapidly with increasing the number $n - n_0$. 'Tunnelling 1' gives too small a result, for example of roughly one order of magnitude already for $n - n_0 = 10$. For $z1 = 40$ and frequencies $\omega = 0.08$ and $\omega = 0.008$, the ionization rate is respectively 1.8 and 3.8 times smaller for 'tunnelling 1' than the exact result. Against the background of this approximation, the asymptotic approximation derived many years ago by Reiss [2, 1] is excellent and approaches all the spectrum very well. The asymptotic approximation slightly overestimates the exact result (obtaining the accuracy of the order of 10% or better) but does not fail with increasing the number $n - n_0$. This is very important because the KFR theory becomes more accurate as $E_n \propto n - n_0$—the kinetic energy of the ionized electron increases. According to the main presumption of the KFR theory, one neglects the interaction with the Coulomb potential in the final state of the outgoing electron.

In figures 3 and 4 we show the SFA ionization rate as a function of intensity for both frequencies considered earlier. It has been still beyond our computational power to use the exact generalized Bessel functions and to calculate ionization rate for very high intensities with sufficient accuracy. However, we have got some numerical experience with not so high intensities. Also the data from figures 1, 2 suggest that the difference between exact and asymptotic results remains small. Therefore in figures 3 and 4 we only compare ionization rates

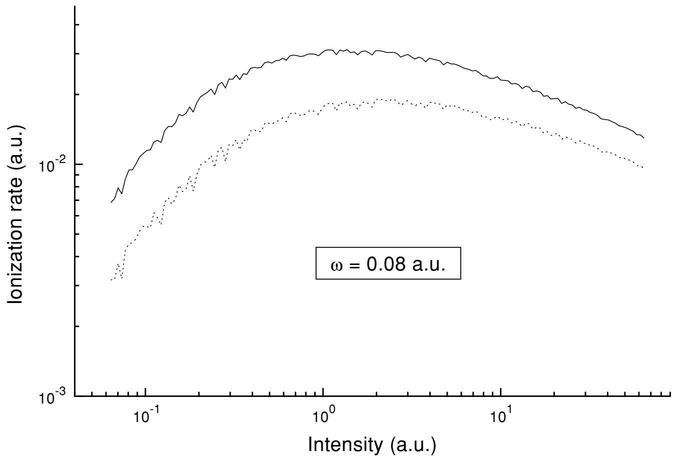

Figure 3. Ionization rate Γ as a function of intensity for $\omega = 0.08$ au. Theoretical SFA calculations were done for the ground state of the hydrogen atom by a linearly polarized plane wave laser field; solid line: with the asymptotic approximation of the generalized Bessel functions and dotted line: with the 'tunnelling 1' approximation of the generalized Bessel functions. The range of intensities corresponds to $10 \leqslant z_1 \leqslant 10^4$ here.

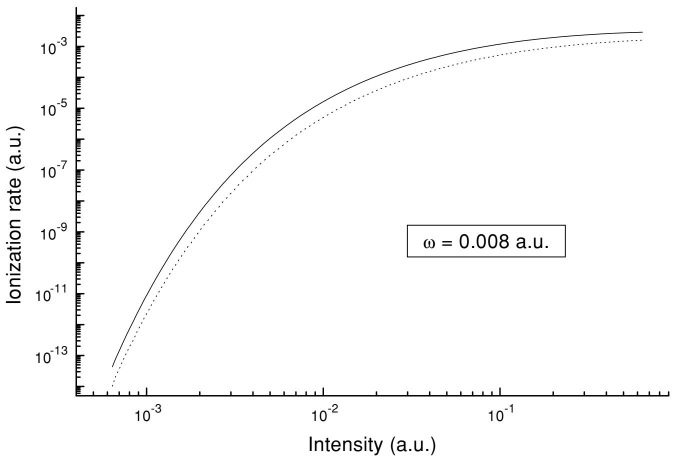

Figure 4. As figure 3, but for a lower frequency $\omega = 0.008$.

in the asymptotic approximation (solid line) with these rates in the 'tunnelling 1' approximation (dotted line). Our intensities cover the whole applicability range of the nonrelativistic SFA and correspond to $10 \leqslant z_1 \leqslant 10^4$. We conclude that 'tunnelling 1' may serve as the useful bottom limitation of ionization rate, even for very high intensities, when calculations involving numerical integration over ϑ are very time-consuming. The integral over ϑ in equation (1) can be calculated analytically in the 'tunnelling 1' approximation, using equation (34) of [1]. Our data have been obtained in this way. We have not investigated the 'tunnelling 2' approximation [1], because it obviously may not be better than 'tunnelling 1', which is already a pretty large simplification.

Acknowledgments

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